Paper&Pencil Skills in the 21st Century, a Dichotomy?

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Abstract There is a worldwide development, better to say a non-development: We teach paper & pencil skills in primary schools almost like we did 30 or 50 or 100 years ago. Till today the primary school teachers spend up to more than 100 hours in the class room to teach and to train old fashioned algorithms though in daily life situations and for business purposes everybody uses a calculator. Why do we waste so much time of our children to teach them things which later on they will not need? We see an emotional dichotomy. Despite the research results from many research projects in many countries there still is the fear that the use of calculators in primary grades will harm mental arithmetic and estimation skills. To explain and to overcome that fear we will reflect the nature of number sense and of paper&pencil skills more carefully. We realize that the development of number sense is an intuitive and unconscious mental process while the ability to get an exact calculation result is trained logically and consciously. To overcome the above dichotomy we must solve the hidden dichotomy ***number sense*** versus ***precise calculation result***. We need a new balance. Different types of examples will be given how we can further the development of number sense in a technology dominated curriculum.

A. Analysis

Specialists from many research projects know that the use of calculators in primary grades does not necessarily harm mental arithmetic and estimation skills. But these "logical" arguments do not count. There still remains an emotional component against the calculator use which cannot be eliminated logically. Thus it is too simple just to claim to replace paper&pencil skills through the calculator. We need more than the ability to get a quick and exact calculation result. To remain mentally independent from the calculator we also must concentrate on automatic mental arithmetic and estimation skills. How do these skills develop? And which changes will we get when we change from paper&pencil skills to calculators?

A more profound view of mathematics learning is necessary to identify the nature of number sense and of paper&pencil skills. Learning and understanding mathematics is based on two different types of mental processes, on logical and conscious arguments as well as on intuitive and unconscious mental processes[1]. These two systems of internal representations produce the two interfering concepts ***precise calculation result*** vs. ***number sense.***

1. Precise Calculation Results

There are three techniques to get precise calculation results: Paper&pencil techniques, using a calculator or computer, and mental arithmetic. The teaching and training of paper&pencil skills is time consuming and the results are less safe than via pressing the appropriate calculator keys. It is obvious why the calculator technique dominates outside from school.

2. Mental Arithmetic

Mental arithmetic is a challenge for teachers to "teach" and for students to "learn" because of the two mental modes which are involved. On the one hand the students should be able to explain ***logically and analytically*** how they get the result. But on the other hand we also expect that for specific problems they can ***react immediately*** in a stimulus response style (stimulus response knowledge for e.g. 1+1 table and 1×1 table).

Furthermore we expect such an ***unconscious and intuitive*** stimulus response reaction also when the student gets confronted with computation mistakes. Either he/she spontaneously notices a conflict with his/her intuitive individual stimulus response knowledge or there is a spontaneous reaction like "this is too big" or "this is too small". The latter describes a conflict between the computation result and the individual personal experiences.

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[1] e.g. Vygotzki talks about spontaneous and scientific concepts, Ginsburg compares informal work and written work, or Strauss discusses a common sense knowledge vs. a cultural knowledge. Strauss (1982) especially has pointed out that these two types of knowledge are quite different by nature, that they develop quite differently, and that sometimes they interfere and conflict ("U-shaped" behavior). For more details see also the web site Meissner / Diephaus (2009).

3. Estimation Skills

Estimation is a challenging activity. Before starting computing we ask for the approximate result of a possible solution. Either the computation task already is given in the classical mathematical symbolic notation or we have to solve a word problem. For the first type of problems the estimation result can be found more easily. Here we must round the numbers and compute with rounded numbers. Estimation in this case is a special analytical and logical approximation technique (in German *Ueberschlagen*).

For word problems usually we first analyze the situation described. We then need a modeling process to get a "translation" of the word problem situation into a mathematical notation of a computation problem where we then can get an estimation result via approximation. But there is an alternative strategy to estimate the result for a word problem.

Analyzing a word problem can and should stimulate also subjective domains of individual experiences related to the situation given (Subjektive Erfahrungsbereiche, cf. Bauersfeld 1983). Intuitively and spontaneously non mathematical knowledge and personal experiences get stimulated, too. Estimation then may become a spontaneous and intuitive reaction like "Oh, this must be about ….".

4. Estimation and Sachrechnen

To estimate spontaneously and intuitively an approximate result for a given word problem we need special experiences, environmental and daily life experiences and experiences in comparing and measuring objects. To develop these experiences the German arithmetic curricula include a special topic called *Sachrechnen* (aspects of environmental and domestic sciences). In Sachrechnen we *compare* objects according to their length, time, weight, etc. (*direkter / indirekter Vergleich* in German) and we *measure* objects: Select a unit und try how often that unit fits into the object[2]. Estimation in Sachrechnen then is quite a different mental activity, it is the internalized process of *comparing* or *measuring* (*Schaetzen* in German).

5. Concept of Numbers

In traditional German curricula for primary schools we introduce step by step the "number spaces" [0 - 20], [0 - 100], [0 - 1.000], and [0 - 1.000.000]. Thus also step by step, the "object number" gets reduced into a sequence of digits and the computation with big numbers gets reduced into manipulations with sequences of digits[3]. Outside from school numbers have a different meaning. Here a number mainly is a measurement number (*Groesse* in German) which describes the size (value, magnitude, …) of an object. It consists of two parts, a *quantity number* and the appropriate *unit* like 345 km or 2830 hours or 562048 cents. The quantity number (*Masszahl* in German) tells us how many units we need to represent the size of that object.

6. Number Sense

We have summarized important aspects which are touched when we talk about number sense: "Number sense refers to an intuitive feeling for numbers and their various uses and interpretations; an appreciation for various levels of accuracy when figuring; the ability to detect arithmetical errors, and a common sense approach to using numbers. ... Above all, number sense is characterized by a desire to make sense of numerical situations" (Reys 1991).

Number sense not only refers to numbers but also to both, to conscious and to unconscious techniques to manipulate numbers, and it also includes a feeling about possible outcomes of these techniques. With a good number sense we roughly can predict the result of calculations, sometimes spontaneously (intuitively) and sometimes consciously (by approximating). Number sense also includes an intuitive feeling for additive and multiplicative structures. A central question for future curricula must be, if we can develop a more effective number sense by the use of calculators than we momentarily do in our traditional curricula.

B. Calculators and Arithmetic Learning

We will start with a warning. An unreflecting use of calculators in primary schools may damage some of the traditional goals of arithmetic education. The uncontrolled use might provoke two problems:

- Pressing keys is so easy. Why shall I still learn mental arithmetic?
- Pressing keys is so safe. Why shall I still control my calculator result?

[2] Sachrechnen also includes the topic money and problem solving activities (problems from real life situations like shopping, planning an excursion, constructing a bird-cage, etc.).

[3] Replacing paper&pencil skills through the use of calculators would not change this view.

Of course, a calculator curriculum must face these problems. But in this paper we will not discuss the PROs and CONs about the use of calculators in primary schools and how paper&pencil techniques could be replaced by a calculator use.

Here we will reflect how arithmetic teaching in primary schools may benefit from the use of calculators and how the use of calculators may help to further the traditional mathematical goals. Of course, the calculator is an excellent tool to get quick and safe calculation results. But besides this property it also may serve as a didactical tool to stimulate intuitive and spontaneous ideas and activities in the teaching and learning processes. The possibility to handle a big bunch of quick calculations without any efforts allows a new working style in the class room which was not possible without calculators or computers. We will summarize and analyze some activities.

7. Stimulus Response Learning

Calculators allow and facilitate stimulus response learning. This can be used in competitions to train mental arithmetic. The basic idea is to compute very quickly a given calculation problem to get then an immediate feedback: *correct* or *wrong*. We developed several exercises starting with problems from the 1+1 and 1×1 table:

(a) Individual work sheets, individual training: Type the problem into the calculator and calculate the result in your head. Then press the "=" and see if you were right. If YES write down the result, if NO do the next problem. Later on work on the still open problems.

(b) Competitions mental computation versus calculator, who is first? At the beginning each student wanted to be in the calculator group, later on almost nobody wanted to be there because "I am quicker in head".

(c) Each student gets a work sheet, the use of calculators is allowed. Who has finished the work sheet first? There may be different work sheets according to the students' abilities.

8. Operators

Calculators with a constant facility[4] allow developing a feeling for additive and multiplicative structures. We hide an operator ⊗k and others must find out which operator we hid:

$$X \xrightarrow{\otimes k} Y$$

Select a value for X, press the calculator keys, and interpret Y. Guess what ⊗k might be. If necessary select another value for X, etc. Finally select additional values for X and predict the results.

9. Calculator Games

There are several calculator games which use the constant facility to detect numbers and operations and to develop a feeling for additive or multiplicative structures. We will present an example with the calculator game *Hit the Target*[5].

Example:

$$? \xrightarrow{x\,17} [800,801]$$

input for z	*display*

Hit the Target

Find via guess and test a number z that z×17 is in the interval [800,801]. Write a protocol of your guesses.

More general: An interval [a,b] is given and a factor k. Find a second number z via guess and test that the product "z×k" is in the interval [a,b].

For primary schools we suggest to concentrate on integers k,z < 100.

Our more than 1000 guess-and-test protocols show that the students after a certain training develop excellent estimation skills. They guess a very good starting number and they develop an excellent proportional feeling. For more details see Meissner 1987.

10. One-Way-Principle

Guess and test or trial and error are not considered to be a valuable mathematical behavior in mathematics education class rooms. But these components are necessary to develop spontaneous and intuitive ideas. Our experiences show that a systematic use of guess and test activities enriches creative and flexible thinking. So we developed a specific teaching method called One-Way-Principle

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[4] These calculators can be "programmed" to work as operators "⊗k". ⊗ stands for the four basic operations.

[5] For more details see Lange / Meissner (1980) and Lange (1984).

(Meissner 2003). The One-Way-Principle is a method to use calculators or computers to explore intuitively and/or consciously many functional relationships of the type

$X \xrightarrow{\sigma} Y$ or in case of the four basic operations $\otimes$ $X \xrightarrow{\otimes k} Y$.

The basic idea of the One-Way-Principle is not to use reverse functions or algebraic transformations but to see the set of variables as a "unit" which gets explored via guess and test.

Concentrating on the four basic operations in primary schools we can explore with simple calculators additive or multiplicative structures of the type "a ⊗ b = c". Here the One-Way-Principle implies not to switch from addition to subtraction (or vice versa) or from multiplication to division (or vice versa). Instead we have to guess "a" (or "⊗" or b or "⊗b") to use then again the originally given key stroke sequence. Independent which variables are given and which are wanted, there is only the ONE WAY to solve all problems: Always use the same simple key stroke sequence of your calculator. The goal for the learner in the guess and test work is to discover intuitively the hidden relations between the variables and to develop a feeling how to get a good first guess (estimation) and how to reach a given target with only a few more guesses (additive resp. proportional feeling). Thus applying the One-Way-Principle furthers some of the intuitive and unconscious skills described above in no. 2 and 3.

C. Reducing Paper&Pencil Techniques

Again, in this paper we will not discuss how paper&pencil techniques could be replaced by a calculator use. But we will reflect how the traditional teaching and training of paper&pencil skills could be reduced. We think the main question is not how to calculate all possible sequences of digits but to ask first for the importance of each technique.

11. Expanding Mental Arithmetic

Mental computations usually are done with small numbers. We suggest to expand the meaning of "small" and to concentrate the four basic operations "a ⊗ b =" on all a and b where a and b are one-digit- or two-digit-numbers. Adding and subtracting two-digit-numbers already is part of traditional curricula. For the multiplication of two-digit-numbers let the students themselves invent appropriate techniques. Paper and pencil should be allowed to write down results from intermediate steps.

12. Proportional Feeling

In parallel to the conscious techniques from no. 11 the students also should get the opportunity to develop an intuitive feeling for possible results. Playing Hit the Target would be an excellent addendum, see no. 9. The students even might select themselves appropriate numbers for Hit the Target (small or big intervals [a,b], no integer solution for z, …).

13. "Large" Numbers

"Large" numbers in this paper are integers with at least 3 digits. Most of these multi digit numbers are unimportant in daily life because we prefer rounded numbers[6]. Putting important rounded numbers on the number line we do not get an equidistant pattern but a pattern which looks more like a logarithmic pattern. It seems as if we determine the importance of numbers in a similar way as we perceive the intensity of light or of sounds (Weber-Fechner law). This would mean that especially in large number spaces there are only a few "important" numbers. The larger the number space is the more unimportant numbers it will have. Do we still need for all these unimportant numbers the traditional paper & pencil techniques? We suggest to concentrate only on the calculating with "important" numbers.

14. Calculating with Rounded Numbers

Rounded numbers are similar to measurement numbers (*Groessen*, see no. 4 and 5). They consist of two parts, a one or two digit quantity number (*Masszahl*) and a unit ("thousands", "millions", etc.). To calculate with rounded numbers we can separate the two parts. We then can calculate with one or two digit numbers and apply techniques about what to do with the units. This approach also furthers Sachrechnen goals:

- For addition and subtraction both numbers must have the same "unit".

[6] e.g. for the size of a swimming pool or a garbage container, for the distance between two cities or between the earth and the moon, or for the weight of an elephant or a lion, etc.

- Changing the unit implies also to convert the related quantity number[7].
- For multiplication and division there are easy rules how to compute with the units. The students themselves might discover these rules.

15. Number Spaces

Reflecting the topics from above we also should rethink the concept of introducing numbers. It is fine to start in the first grade with [0 - 20] and then [0 - 100]. But when we start using calculators the number space suddenly gets unlimited. We need a spiral approach in which the students themselves can discover numbers and number properties in individual own subjective domains of experiences and where they then can discuss their experiences. A spiral approach also would help to develop a much broader number sense.

16. Decimal Numbers

When we introduce calculators in primary schools we must be aware that the students very soon will discover decimal numbers in the display. But they already have a basic knowledge of writing decimals. According to our experiences they are just happy to learn that 23.5 can be interpreted as 23 cm and 5 mm, or 12.69 as 12 $ and 69 ct or 3.125 as 3 km and 125 m. And when there are more digits behind the “point”? Usually the children accept the simple answer “just ignore those digits” which corresponds to the view from above to distinguish between important and unimportant numbers.

17. FORUM

There is an internet web site to continue the discussion about the future of paper and pencil skills. Those who are interested to offer own contributions to that web site kindly are asked to write an email to Hartwig Meissner (meissne@uni-muenster.de). The FORUM web address is: http://wwwmath.uni-muenster.de/didaktik/u/meissne/WWW/Forum-P&P.htm

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Meissner, H. (2007): Primary School - Calculators or Paper & Pencil Techniques? Web: http://wwwmath.uni-muenster.de/didaktik/u/meissne/WWW/mei146.doc

Meissner, H. (2008 ff): FORUM on the ***Future of Paper & Pencil Skills***. Web: http://wwwmath.uni-muenster.de/didaktik/u/meissne/WWW/Forum-P&P.htm

Meissner, H.; Diephaus, A. (2009): The Development of Number Sense. Web: http://wwwmath.uni-muenster.de/didaktik/u/meissne/WWW/mei150.doc

[7] Getting experiences in changing units also may further the development of spontaneous and intuitive reactions as described in no. 3.